MATH 590: QUIZ 6 SOLUTIONS

Name:

1. State five of the equivalent conditions for the matrix A to be invertible, as given in the Daily Update of March 1. (5 points)

Solution. See the Daily Update of March 1.

2. Find an orthonormal basis consisting of eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Be sure to check that your basis is orthonormal. (5 points)

Solution. The eigenvalues of A are the roots of the polynomial

$$\det \begin{pmatrix} x-1 & -2 \\ -2 & x-1 \end{pmatrix} = (x-1)^2 - 4 = x^2 - 2x - 3 = (x-3)(x+1),$$

and thus are 3 and -1.

The eigenspace of 3 is the nullspace of the matrix $\begin{pmatrix} 3-1 & -2 \\ -2 & 3-1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, which row reduces to $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. The nullspace of this last matrix is clearly $\{\begin{pmatrix} t \\ t \end{pmatrix} \mid t \in \mathbb{R}\}$, which has basis $v_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenspace of -1 is the nullspace of the matrix $\begin{pmatrix} -1-1 & -1 \\ -2 & -1-1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$ which row reduces to $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. The nullspace of this last matrix is clearly $\{\begin{pmatrix} t \\ -t \end{pmatrix} \mid t \in \mathbb{R}\}$, which has basis $v_2 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Normalizing v_1 and v_2, we get $u_1 := \frac{1}{\sqrt{2}} v_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ and $u_2 := \frac{1}{\sqrt{2}} v_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$. Moreover, since $\begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} \neq 0$, u_1, u_2 form a basis for $\mathbb{R}^2$, and $u_1 \cdot u_2 = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot (-\frac{\sqrt{2}}{2}) = 0$, showing that u_1, u_2 form an orthonormal basis consisting of eigenvectors for A.